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NON-LINEAR UNSTEADY CREEP OF AN ICE SHEET ON A HYDRAULIC FOUNDATION†

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The problem of the non-linear unsteady creep of the bending deformation of an ice sheet which partially covers a hydraulic foundation is considered within the framework of the hypothesis of plane sections. In a plan view the sheet is a strip of finite width and length with a single clamped end. This may be shore ice close to the wall of a hydroelectric structure or a plate which has been specially sawn out in an ice sheet for natural experimental investigation. A frequently adopted relationship [1, 2] between the strain, creep and stress which is, to some degree, under the sign of a Volterra-type time operator with a non-difference kernel is used to describe the rheology of the ice. A non-linear integro-differential equation for the bending moment in a sheet on a hydraulic foundation is obtained which is solved by expansion in a series in a certain small time parameter and, subsequently, numerically along a coordinate by the monotonic sweep method. The deflection of the sheet is also found. Characteristic cases of the change in the bending moment and the deflection along the length of the sheet with time are considered. Copyright © 1996 Elsevier Science Ltd.

1. Let a sheet of ice of thickness h, width $l > h$ and length $L > l$ lie on a hydraulic foundation which we shall simulate with a Fuss–Winkler foundation [3] with a bedding parameter $k = \rho g$, where ρ is the density of the liquid and g is the acceleration due to gravity. One of the ends of the sheet is clamped along its length and the other is loaded with a transverse force F and a bending moment G (Fig. 1). We shall assume that the sheet does not become detached from the foundation under the action of these loads.

Natural ice is a solid with an extremely complex multiphase structure [4]. Apparently, it is impossible to describe its deformation, as a continuous medium, using a single universal law. Here, it is proposed, in connection with the study of the non-linear unsteady creep of the flexural deformation of an ice sheet, in the case of a uniaxial stressed state, to use the relation [1, 2]

$$\varepsilon = \frac{\sigma}{E} + \int_{\tau_*}^{t} B(t,\tau)\,\sigma|\sigma|^{m-1}d\tau \tag{1.1}$$

where ε and σ are the strain and the stress, E is Young's modulus, t is the time, τ_* is the moment of the load, $B(t, \tau)$ and m are a function (the creep kernel) and the non-linearity parameter ($\min_{t,\tau} B(t, \tau) > 0$ and $m > 1$) determined experimentally. It follows from (1.1) that the instantaneous deformation of the ice is assumed to be elastic. In order to improve the accuracy of the model (1.1), E is permitted to depend on t.

The relative elongation of a filament of the sheet, located in a section at a distance z from the neutral axis is $\varepsilon = z/R$ (R is the radius of curvature of the neutral plane). In the case of small deflection of the sheet w, it is possible to put $\varepsilon = zw_x''$ with sufficient accuracy. When this is taken into account, we can write relation (1.1) in the form

$$Ez\frac{\partial^2 w}{\partial x^2} = \sigma + E\int_{\tau_*}^{t} B(t,\tau)\,\sigma|\sigma|^{m-1}d\tau \tag{1.2}$$

Regarding the dependence of σ on z ($|z| \leq h/2$), we can assert that

$$|\sigma| \sim |z|^{\alpha(|z|)} \tag{1.3}$$

and the function $\alpha(|z|)$ increases monotonically as $|z|$ decreases from a value of $\alpha(h/2) = m^{-1}$ to a value of $\alpha(0) = 1$ (when $E = \infty$, as has been shown in [5], $\alpha = m^{-1}$).

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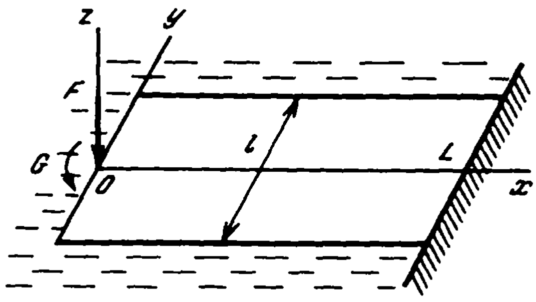

Fig. 1.

Let Ω be the area of the cross-section of the sheet. This is a rectangle with sides h and l. On multiplying (1.2) by $z\,d\Omega$ and integrating over the cross-section, we find

$$EI\frac{\partial^2 w}{\partial x^2} = M + E\int_{\tau_0}^{t} B(t,\tau)\,d\tau \int_{\Omega} \sigma|\sigma|^{m-1} z\,d\Omega \tag{1.4}$$

$$\left(I = \int_{\Omega} z^2 d\Omega = \frac{lh^3}{12},\quad M = \int_{\Omega} \sigma z\,d\Omega\right)$$

where I is the moment of inertia of the cross-section and M is the bending moment in a certain cross-section of the sheet. Since, in some cross-section or other, σ and z always have the same sign or opposite signs, the inner integral in (1.4) is equal either to J or to $-J$, where

$$J = \int_{\Omega} |\sigma|^m |z|\,d\Omega \tag{1.5}$$

Let $m^{-1} + n^{-1} = 1$. Then, by virtue of Hölder's inequality [6]

$$|M| = \int_{\Omega} |\sigma||z|\,d\Omega \leqslant C_1\left(\int_{\Omega} |\sigma|^m |z|\,d\Omega\right)^{1/m} \tag{1.6}$$

$$\frac{1}{C_1^m}|M|^m \leqslant J,\quad \frac{1}{C_1^m} = \left(\frac{4}{h^2 l}\right)^{m-1}$$

Hence, a lower estimate has been obtained for integral (1.6).

We shall use Favard's inequality [6] to obtain an upper estimate

$$\frac{1}{b-a}\int_a^b [f(x)]^p dx \leqslant \frac{2^p}{p+1}\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right]^p \tag{1.7}$$

where $p > 1$ and $f(x)$ is a non-negative continuous convex ($f'(x) < 0$) function in $[a, b]$ (Favard's inequality is given in [7] with an error which subsequently has an effect on the accuracy of the upper estimate of the integral (1.5). The corrected result is given here.) We note that

$$\int_{\Omega} |\sigma|^m |z|\,d\Omega = l\int_0^{\omega} |\sigma|^m dz_1 \quad \left(\omega = \frac{h^2}{4}\right) \tag{1.8}$$

where $m > 1$ and $|\sigma|$, like the function z_1, is convex by virtue of (1.3). Then, from (1.7) we have

$$J \leqslant \frac{1}{C_2^m}|M|^m,\quad \frac{1}{C_2^m} = \frac{2^m}{m+1}\left(\frac{4}{h^2 l}\right)^{m-1} \tag{1.9}$$

So, when account is taken of relations (1.6) and (1.9), we can approximately put

$$J \approx C^{*}|M|^{m}, \quad C^{*} = (C_1^{-m} + C_2^{-m})/2 \tag{1.10}$$

When $1 \leqslant m \leqslant 3$ we have $1 \leqslant C^{*}/C_1^{-m} \leqslant 1.5$; $1 \leqslant C^{*}/C_2^{-m} \leqslant 0.75$. On substituting expression (1.10) into (1.4), we obtain the following approximate equation for describing the flexural deformation of an ice sheet under conditions of non-linear unsteady creep

$$EI\frac{\partial^2 w}{\partial x^2} = M + C^{*}E\int_{\tau_*}^{t} B(t,\tau)\,M|M|^{m-1}d\tau \tag{1.11}$$

Note that an equation similar to (1.11) was obtained without proof in [5].

2. From the side of the hydraulic foundation, a load $q(x, t) = klw(x, t)$ which is distributed along the length acts on the sheet. At the same time, we know [3] that $\partial^2 M/\partial x^2 = -q(x, t)$, and therefore

$$klw(x, t) = -\partial^2 M/\partial x^2 \tag{2.1}$$

Substituting (2.1) into (1.11), we obtain an integro-differential equation for the bending moment $M(x, t)$

$$\frac{\partial^4 M}{\partial x^4} + \frac{kl}{EI}M + \frac{klC^{*}}{I}\int_{\tau_*}^{t} B(t,\tau)\,M|M|^{m-1}d\tau = 0 \tag{2.2}$$

The boundary conditions are

$$M = -G, \quad M' = -F \quad (x = 0) \tag{2.3}$$

$$w = w' = 0 \quad (x = L) \tag{2.4}$$

By virtue of (2.1), the latter conditions can also be written as follows:

$$M'' = M''' = 0 \quad (x = L) \tag{2.5}$$

Thus we have Eq. (2.2) with boundary conditions (2.3) and (2.5). By solving it, we can then find w using formula (2.1).

For a sufficiently old ice sheet the creep kernel $B(t, \tau)$ may be assumed to be a difference kernel [1] and, on the basis of experimental data [4], we can assume that

$$B(t - \tau) = B_\infty + B_0\exp[-\mu(t - \tau)] \tag{2.6}$$

Allowing for (2.6), we change in (2.2) to the reduced time

$$\zeta = 1 - e^{-\mu t}\ (\zeta \in [0, 1)), \quad \theta = 1 - e^{-\mu\tau} \quad (\theta \in [0, 1)) \tag{2.7}$$

Equation (2.2) then takes the form

$$\frac{\partial^4 M}{\partial x^4} + \frac{kl}{EI}M + \frac{klC^{*}}{I\mu}\left[B_\infty\int_0^{\zeta}\frac{M|M|^{m-1}}{1-\theta}d\theta + B_0(1-\zeta)\int_0^{\zeta}\frac{M|M|^{m-1}}{(1-\theta)^2}d\theta\right] = 0 \tag{2.8}$$

We shall seek a solution of Eq. (2.8) in the form of the series

$$M(x,\zeta) = \sum_{i=0}^{\infty} M_i(x)\,\zeta^i \tag{2.9}$$

On substituting (2.9) into (2.8), carrying out a number of transformations and equating the expressions in like powers of ζ, we arrive at the following system of successively solvable linear differential equations

$$M_0^{(4)} + J_1 M_0 = 0$$
$$M_i^{(4)} + J_1 M_i = -(J_2 S_i + J_3 T_i) \quad (i \geq 1) \tag{2.10}$$

Here

$$J_1 = \frac{kl}{EI}, \quad J_2 = \frac{klC^* B_\infty}{I\mu}, \quad J_3 = \frac{klC^* B_0}{I\mu}$$
$$S_i = \frac{1}{i}\sum_{m=0}^{i-1} f_m \quad (i \geq 1), \quad T_1 = f_0 \tag{2.11}$$
$$T_i = \frac{1}{i(i-1)}\sum_{m=1}^{i-1} m f_m \quad (i \geq 2)$$

Expressions for f_m ($m = 0, 1, \ldots, 4$) are given in [7]. It is characteristic that f_m depends on all M_0, $M_1, \ldots, M_{m-1}$. The boundary conditions for Eqs (2.10) have the form

$$M_0 = -G, \quad M_0' = -F \quad (x = 0), \quad M_0'' = M_0''' = 0 \quad (x = L)$$
$$M_i = M_i' = 0 \quad (x = 0, i \geq 1), \quad M_i'' = M_i''' = 0 \quad (x = L, i \geq 1) \tag{2.12}$$

For M_0 and w_0, we have the classical elastic-instantaneous solution

$$M_0 = e^{\beta x}(A_0 \cos\beta x + B_0 \sin\beta x) + e^{-\beta x}(C_0 \cos\beta x + D_0 \sin\beta x)$$
$$w_0 = -\frac{2\beta^2}{k}[e^{\beta x}(B_0 \cos\beta x - A_0 \sin\beta x) - e^{-\beta x}(D_0 \cos\beta x - C_0 \sin\beta x)] \tag{2.13}$$

$$A_0 = -G - C_0, \quad C_0 = \frac{1}{2}\left[B_0 + D_0 + \frac{F}{\beta} - G\right]$$

$$B_0 = \left\{\left[-\frac{F}{\beta} - G + e^{-2\beta L}\left(-\frac{F}{\beta} + G\right)\right]\sin\beta L + D_0[2e^{-2\beta L}\cos\beta L - (1 + e^{-2\beta L})\sin\beta L]\right\}\times$$
$$\times [2\cos\beta L + (1 + e^{-2\beta L})\sin\beta L]^{-1}$$

$$D_0 = \left[-\frac{F}{\beta} - G + e^{-2\beta L}(1 - 2\cos\beta L \sin\beta L)\left(-\frac{F}{\beta} + G\right) - 2Ge^{-2\beta L}\sin^2\beta L\right]\times$$
$$\times [4e^{-2\beta L}\cos^2\beta L + (1 + e^{-2\beta L})^2]^{-1}$$

The functions M_i and w_i, when $i \geq 1$, were found numerically using the standard scheme which arises in the approximation of boundary-value problems for fourth-order ordinary differential equations. The methods of Gaussian elimination and an algorithm for monotonic sweep [8] were used. The program was written in C language.

3. To illustrate the problem, we will now present some examples. The following data, based on experimental material [4], were inputted into the program: $E = 4 \times 10^9$ kg/ms^2; $B_0 = 84 \times 10^{-8}$ (ms^2/kg)^{m}s^{-1}; $B_\infty = 5.6 \times 10^{-8}$ (ms^2/kg)^{m}s^{-1}; $\mu = 3 \times 10^{-2}$ s^{-1}, $m = 1.72$. In addition, we took $\rho = 10^3$ kg/m^3; $g = 9.81$ m/s^2; $h = 0.25$ m; $l = 1$ m; $G = 0$ kg^2/s^2; $F = 500$ kg/s^2. All of the results subsequently presented are for the case when sixth-order terms were retained in (2.9). Here, it was numerically established that this number of terms is sufficient for $t \sim 200$ s.

1. Let the sheet have a finite length $L = 10$ m. Curves for the bending moment M (kg^2/s^2) and the deflection w (m) as a function of the change in the x coordinate at fixed values of the time are shown in Fig. 2. Curves 1–4 correspond to time $t = 0$, 30, 60 and 200 s. Curves 1 illustrate the elastic-instantaneous solution of the problem and curves 4 correspond to the solution having reached a steady state. All the curves increase monotonically. It is most probable that the sheet breaks at the clamped end, that is, at $x = 10$ m.

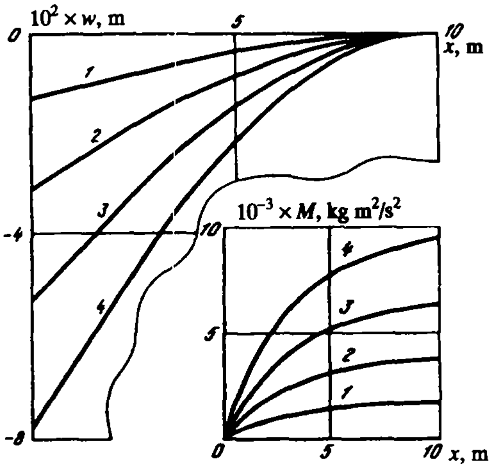

Fig. 2.

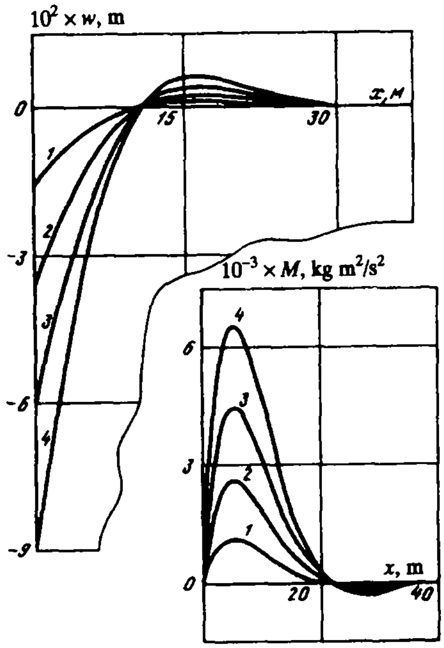

Fig. 3.

2. Let the sheet be semi-infinite. As above, the curves for M(kg m^2/s^2) and w (m) are shown in Fig. 3 as a function of x at fixed instants of time which are the same as those in Fig. 2. The form of all the curves changes and they are not monotonic. The maximum bending moment is attained at $x = 5.6$ m. This is the most probable point for the sheet to fracture.

3. Let the sheet be semi-infinite and heavy. The picture will then be similar to that shown in Fig. 3 with the sole difference that $w \to h\rho_+/\rho$ when $x \to \infty$. Here, ρ_* is the density of ice.

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